

MA 323 Geometric Modelling

Course Notes: Day 19

Differential Geometry of Plane Curves and G^1 Bezier Splines

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We have considered over the last couple of days, C^0 , C^1 and C^2 Bezier splines, that is composite Bezier curves (piecewise defined curves with component arcs as Bezier curves) that have functional continuity. Today, we want to weaken these conditions to pure geometric information, as the specification of C^k curves requires the use of the parameterization of the curve to place the control points for the requisite Bezier curves. As most of you probably noticed and were to show in some of the exercises, one can obtain a C^0 Bezier spline that looks smooth by placing the control points p_{3i} anywhere on the line segments $p_{3i-1}p_{3i+1}$, not necessarily at the midpoint as required for a C^1 Bezier spline. The point p_{3i} does have to be between p_{3i-1} and p_{3i+1} to appear smooth. Such curves can be shown to be parametrically smooth by choosing the appropriate non-uniform knot sequence, as you were requested to do in the exercises. However, it is more convenient to view the construction as constructing splines that are geometrically smooth, G^k Bezier splines.

Theoretically, to state results and understand the significance of the choice of a non-uniform knot sequence in geometric terms using only the local parameters for each individual Bezier curve instead of the global parameter of the spline, it is useful to look at the geometry of curves. This means the differential geometry of curves; geometric results obtained by the use of calculus. In addition, to understanding the construction of G^k splines, understanding the differential geometry of curves will yield other geometric methods for measuring closeness.

19.1 Differential Geometry of Plane Curves

In this section, we discuss the geometry of plane curves, and use the geometry of plane curves to judge the “goodness” of an approximation of a curve by a Bezier Curve and a Bezier Spline. The geometry of a plane curve means the curvature of a plane curve, as the fundamental theorem of plane curves states that given a continuous curvature function $\kappa(s)$ there is a unique curve (up to translation and rotation) parametrized by arc length having curvature κ . The fundamental theorem of plane curves also gives us a method for judging whether two curves are close. If the curvature functions are close and the curves are aligned (pass through the same point and have the same tangent). This theorem provides us with a very good mathematical method to judge the closeness of two curves. Curvature is also used in *fairing* curves, that is smoothing curves.

19.2 Geometry of Plane Curves

Given a parametric plane curve $c(t)$ which we interpret as the position of a particle at time t , the first derivative

$$c'(t) = \lim_{\Delta t \rightarrow 0} \frac{c(t + \Delta t) - c(t)}{\Delta t}$$

represents velocity vector of the particle and the second derivative (the derivative of the derivative) represents the acceleration vector of the particle. These are the interpretations of a curve through motion. They mean little geometrically, and for our purposes in this section they are meaningless. This is the difference between a road, and the position of a car on the road. Geometrically, the path of the car is road. But the road itself, does not give the position of the car. Geometrically, and for geometric design purposes, we are interested in the construction of the road, not the position of the car.

The first derivative does provide some geometric information if it is nonzero. The first derivative gives the tangent line to the curve (a geometric interpretation of the derivative). The tangent line is geometrically what we want to consider as the first derivative. More specifically, the first “geometric” derivative is the unit tangent vector $T(t)$ of a curve, which we define as

$$T(t) = \frac{c'(t)}{\|c'(t)\|}$$

where $\|\cdot\|$ is the length of the vector. This requires that we have an orientation of the curve (a direction of measurement or motion), to define the derivative. In fact, the unit tangent vector is just the derivative with respect to arc length, that is

$$T(t) = \lim_{\Delta t \rightarrow 0} \frac{c(t + \Delta t) - c(t)}{\|c(t + \Delta t) - c(t)\|}.$$

or

$$T = \frac{dc}{ds}$$

where s is the arc-length parameter of the curve. The arc-length parameter is defined by computing the arc-length of the segments of the curve. Recall the arc-length of a curve is defined to be

$$s = \lim_{\|\Delta t\| \rightarrow 0} \sum_{i=1}^n \|c(t_i) - c(t_{i-1})\| = \int_a^b \|c'(t)\| dt$$

where $a = t_0 < t_1 < \cdots < t_n = b$ and $\|\Delta t\| = \max(t_1 - t_0, t_2 - t_1, \dots, t_n - t_{n-1})$.

The second “geometric” derivative of a curve is the curvature. The derivative of the unit tangent vector T with respect to arc-length is the curvature vector. To define the curvature (a function), we first choose a consistent normal vector for the curve. Given a tangent line, there is a well defined normal line (a line perpendicular to the tangent line that passes through the given point on the curve - see diagram below). The unit normal vector $N(t)$ to a curve is chosen to be a vector on the normal line such that the vectors T and N form a right-handed basis of the plane. This means that considered as a list of vectors $[T, N]$, the vector $N = (-T_2, T_1)$ where $T = (T_1, T_2)$ in an orthogonal coordinates system. The curvature κ is defined to be the multiplicative factor κ such that

$$\frac{dT}{ds} = \frac{T'(t)}{\|c'(t)\|} = \kappa(t) N(t).$$

We remark that this definition of curvature is different than the standard definition of curvature in calculus. However, the absolute value of the curvature is the reciprocal of the

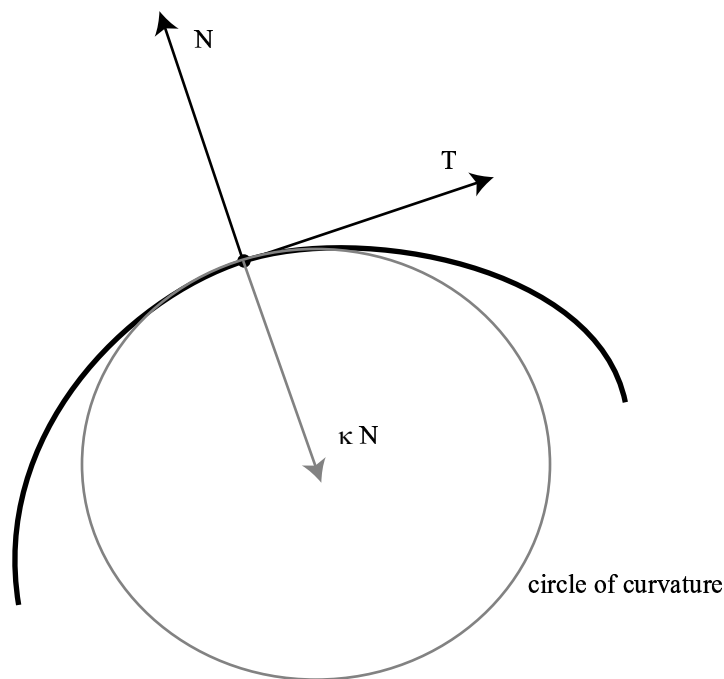


Figure 1: Tangent, Normal, and curvature of a curve

radius of curvature or radius of the best-fit circle to a curve. The sign only tells which direction the curve is turning.

The principal difference between this definition of curvature and the definition that is most calculus textbooks is that the above definition allows the curvature to be signed. In this definition, we have positive curvature and negative curvature. The sign determines the direction of turning. If the curvature is positive then the curve turns in the direction of the unit normal vector and if the curvature is negative then the curve turns in the opposite direction of the unit normal vector, see diagram below. The advantage of this definition is that it allows us to determine a curve solely in terms of its curvature.

19.3 Fundamental Theorem of Plane Curves

The importance of curvature is given by the fundamental theorem of plane curves.

Theorem. *Given a continuous function $\kappa(s)$. There is a unique curve $c(s)$ parameterized with respect to arc-length (unique up to a translation and a rotation) whose curvature is given by κ .*

This result follows directly from solving the system of differential equations $c'(s) = T(s)$, $T'(s) = \kappa(s)N(s)$, and $N'(s) = -\kappa(s)T(s)$. The uniqueness of the curve is given since one can choose an initial point $c(0) = x_0$ and an initial tangent vector $T(0) = v_0$. The choice of initial normal vector $N(0)$ is determined by the choice of initial tangent vector.

Curvature Calculations

Given a parametric representation of a curve $c(t)$, in order to calculate the curvature of c , it is useful to have formulas without using arclength parameters. These calculations involve

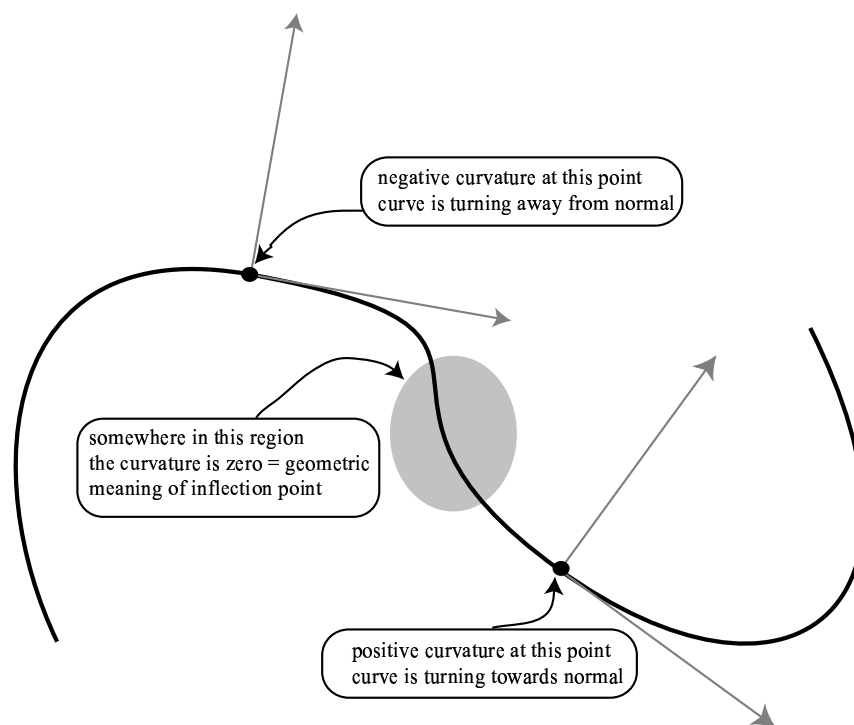


Figure 2: Difference in signs of curvature

differentiation of vector valued function and it is useful to use the standard rules in calculus of vector valued functions to perform these calculations. We will not go into the details of these calculations today. You should review the necessary calculus, which will be used in the following sections. The curvature calculations will be reviewed in the context of the calculations needed calculating the curvature of a Bezier curves.

Closeness of Approximations

The importance of curvature for our purposes is that curvature provides a measure of the closeness of two curves. This means in rough terms if two curves have curvatures (as functions of arclength) that are close then the two curves are approximately the same as long as the curves agree at point and point roughly in the same direction. It is important to note that the curvatures must be compared with respect to arc length in order to get closeness of the two curves.

Previously, we considered two curves to be close if they were close in distance. This meant that the both curves lied within an ϵ neighborhood of each other (see figure below). This is a crude form of closeness for it does not account for the variation of the curve (how the two curves intertwine). To account how the curves intertwine, one needs to have not only the curves close but the tangent vectors close at corresponding points. The next measure of closeness is to have the curvatures close together, the tangent vectors close together and the curvatures close together. The fundamental theorem of plane curves essentially says if the curvatures of two curves are close then the curves are close in distance and in tangents (up to a rotation and translation), which means that if one can base closeness off a single calculations.

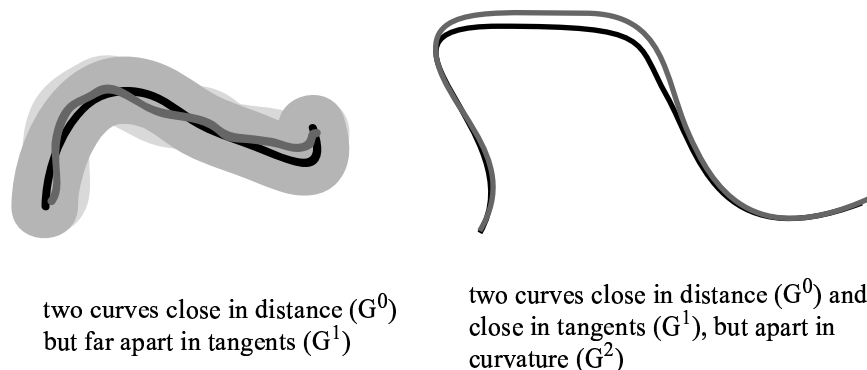


Figure 3: Notions of closeness

Differences Between Notions of Closeness

We will be more specific about these notions of closeness later when we will give complete details in the calculations, and computing whether two curves are close. Today, we want to concern ourselves with the general concepts and the general idea. In the exercises, you are to use the notion of curvature and the ideas of when two curves are close to attempt to approximate curves and create curves.

19.4 Geometric Continuity

In this section, we want to discuss piecewise Bezier curves that are geometrically continuous. A plane curve is G^1 if the tangent line is defined everywhere, meaning the unit tangent vector $T(t)$ is continuous, and a plane curve is G^2 if it is G^1 and the curvature of the curve is continuous. The notation G^k represents geometric continuity rather than the functional continuity represented by C^k . In particular, G^k means k th order geometric continuity rather than C^k which means k th order functional continuity. Recall that functional C^1 continuity means the derivative is well-defined at each joint point. G^1 continuity is slightly weaker and means that the tangent line at each joint point is well-defined. This means that every curve that is C^1 continuous is G^1 provided the unit tangent vector is well-defined, and also every curve that is C^2 will be G^2 provided that the unit tangent vector is well-defined for the curve.

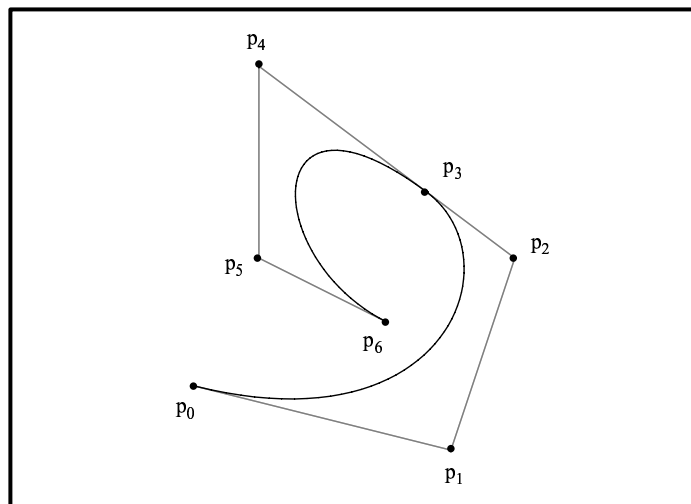
19.5 G^1 Bezier splines

The control points $p_0, p_1, p_2, p_3, p_4, p_5, p_6$ of two cubic Bezier curves that are joined in a G^1 fashion must have p_3 somewhere on the line segment p_2p_4 . The joint point can be any point on the interior of the line segment not just the midpoint. This is because the unit tangent vector at p_3 is equal to

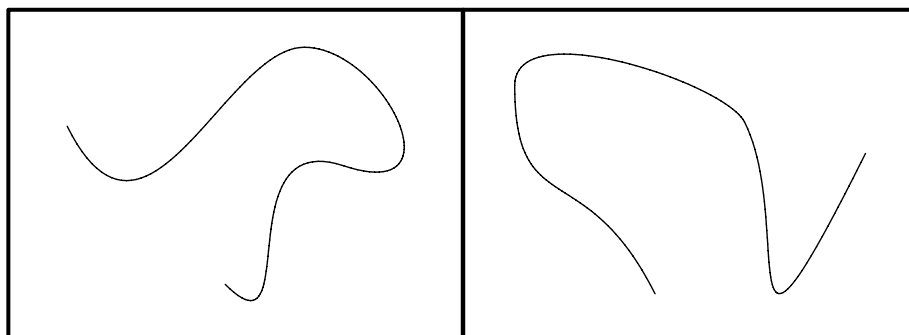
$$\frac{p_3 - p_2}{\|p_3 - p_2\|} = \frac{p_4 - p_3}{\|p_4 - p_3\|}$$

where $\|x - y\|$ is the distance between x and y , which places no restriction on the placement of p_3 other than that it is between p_2 and p_4 . See the diagram below.

It is hard to tell the difference between a C^1 and G^1 spline by looking at them. The difference is that G^1 curves are more flexible, you have extra “degrees of freedom” when designing with G^1 curves rather than with C^1 curves. However, geometrically one can not see the

Figure 4: Joining of two cubic Bezier curves in a G^1 fashion

difference, as you can not see the parameterization of the curves to detect the difference between the curves without knowing the control points. For instance, look at two curves below. Can you tell which is the C^1 curve and which is the G^1 curve?

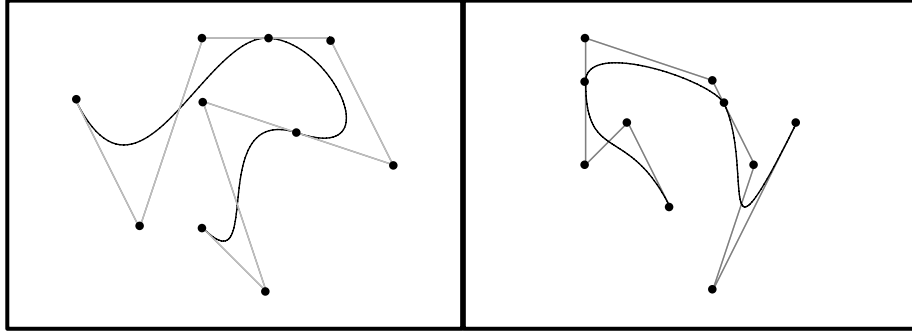
Figure 5: Which is the C^1 Bezier spline and which is the G^1 Bezier spline

The splines are given below with the control points for the piecewise Bezier curves that define them. You should now be able to tell which is the C^1 spline and the G^1 spline, knowing that C^1 curve has the joint points p_{3i} the midpoint of the segments p_{3i-1} and p_{3i+1} .

19.6 Bezier Control Points for a G^1 splines

To create a G^1 spline, we still give $2L + 2$ control points $d_{-1}, d_0, d_1, \dots, d_{2L}$ to define the curve. But now, we also need to provide a method for choosing the joint point. The standard method is to use a *knot sequence* $t_0 < t_1 < \dots < t_L$, a set of $L + 1$ numbers which we use to define the joint points. The control splines for the piecewise Bezier curve is given by first setting

$$p_0 = d_{-1}, \quad p_1 = d_0, \quad p_2 = d_1$$

Figure 6: The C^1 B-spline and G^1 B-spline

and

$$p_{3i+1} = d_{2i}, \quad p_{3i+2} = d_{2i+1}$$

where $i = 1, \dots, L-1$, and

$$p_{3L} = d_{2L}.$$

The joint points p_{3i} , $i = 1, \dots, L-1$ are now given using the knot sequence by setting

$$p_{3i} = \left(\frac{t_{i+1} - t_i}{t_{i+1} - t_{i-1}} \right) p_{3i-1} + \left(\frac{t_i - t_{i-1}}{t_{i+1} - t_{i-1}} \right) p_{3i+1}.$$

The knot sequence is used to specify the ratio of the points p_{3i-1} , p_{3i} , p_{3i+1} , that is the barycentric coordinates of the point p_{3i} in terms of the points p_{3i-1} and p_{3i+1} .

To understand the last displayed equation in terms of barycentric coordinates, it is useful to note that the above formula is designed to show that the numbers $[t_{i-1}, t_i, t_{i+1}]$ on the number line are associated to the points $[p_{3i-1}, p_{3i}, p_{3i+1}]$ so that the barycentric coordinates of t_i written in terms of t_{i-1} and t_{i+1} is equal to the barycentric coordinates of p_{3i} in terms of p_{3i-1} and p_{3i+1} . Noting that

$$(c-b)a + (b-a)c = ca - ba + bc - ac = b(c-a)$$

and therefore

$$\frac{c-b}{c-a}a + \frac{b-a}{c-a}c = b.$$

We thus have

$$\frac{t_{i+1} - t_i}{t_{i+1} - t_{i-1}} t_{i-1} + \frac{t_i - t_{i-1}}{t_{i+1} - t_{i-1}} t_{i+1} = t_i$$

using $a = t_{i-1}$, $b = t_i$, $c = t_{i+1}$.

REMARKS: (1) If the sequence $t_0 < t_1 < t_2 < \dots < t_L$ is uniform, meaning $t_1 - t_0 = t_2 - t_1 = \dots = t_L - t_{L-1}$, then the curve will be C^1 . (2) If you reparameterize the curve with respect to arc-length the curve will be C^1 . This is really what G^1 means, differentiable with respect to the arc-length parameter.

19.7 Functional Representation for a G^1 Spline

The functional representation for a G^1 spline is slightly different than the functional representation for the piecewise Bezier curve with the control points. The knot sequence has a secondary use other than defining the ratios of the joint points. The knot sequence defines

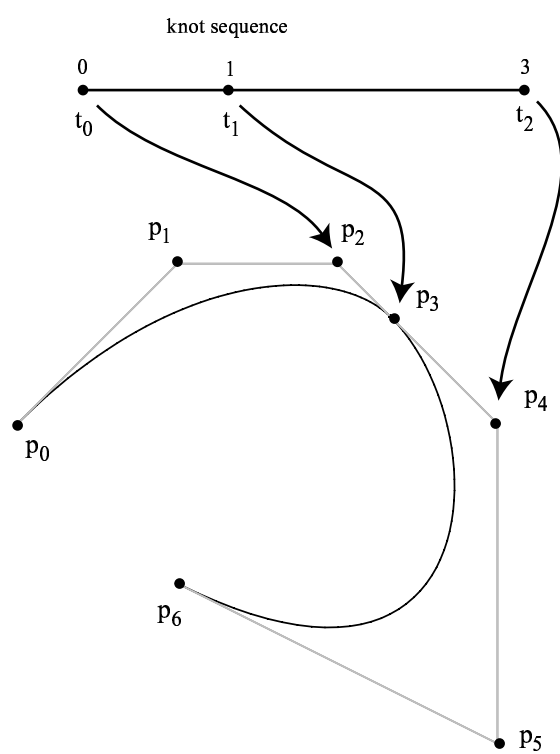


Figure 7: Barycentric equivalence of $[t_{i-1}, t_i, t_{i+1}]$ and $[p_{3i-1}, p_{3i}, p_{3i+1}]$

the values of the parameter for each curve segment. The global parameter for the curve t ranges between t_0 and t_L , so a G^1 spline will be represented by $b : [t_0, t_L] \rightarrow E^n$, and each segment will be defined on an interval $[t_{i-1}, t_i]$.

Let $t_0 < t_1 < t_2 < \dots < t_L$ be the knot sequence and $p_0, p_1, p_2, \dots, p_{3L}$ be the control points for the piecewise Bezier curve. The functional representation is given by defining a local variable for each segment. The local variable for the i th segment is $s = (t - t_{i-1}) / (t_i - t_{i-1})$, which ranges between $0 \leq s \leq 1$. Therefore, in the local variable s , the i th segment is then given by

$$B_0^3(s) p_{3i-3} + B_1^3(s) p_{3i-2} + B_2^3(s) p_{3i-1} + B_3^3(s) p_{3i}.$$

In the global variable t , we have by substituting

$$B_0^3\left(\frac{t-t_{i-1}}{t_i-t_{i-1}}\right) p_{3i-3} + B_1^3\left(\frac{t-t_{i-1}}{t_i-t_{i-1}}\right) p_{3i-2} + B_2^3\left(\frac{t-t_{i-1}}{t_i-t_{i-1}}\right) p_{3i-1} + B_3^3\left(\frac{t-t_{i-1}}{t_i-t_{i-1}}\right) p_{3i}.$$

The global variable t ranges between t_{i-1} and t_i for the i th segment.

19.8 Exercises

1. Complete the interactive exercises associated with the G^1 Bezier spline applet.
2. Given the G^1 control points below, construct the control points for the C^0 Bezier spline using the knot sequences, and then sketch the spline.
 - (a) $t_0 = 0, t_1 = 1, t_2 = 2, t_3 = 5$
 - (b) $t_0 = 0, t_1 = 2, t_2 = 4, t_3 = 6$
 - (c) $t_0 = 0, t_1 = 1, t_2 = 3, t_3 = 4$

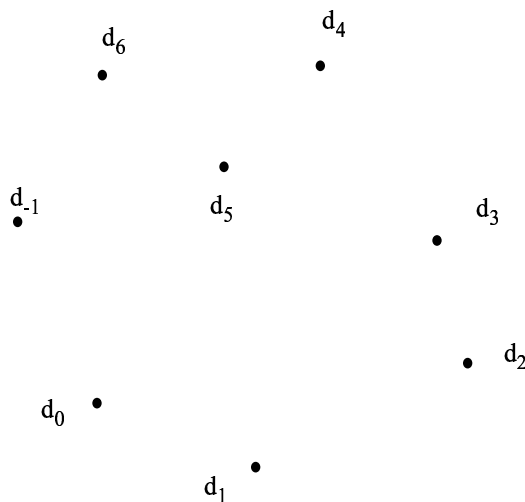


Figure 8: Construct the G^1 spline from these control points

3. Show that with respect to the global parameter t defined through the knot sequence a G^1 Bezier spline is actually C^1 .