

# MA 323 Geometric Modelling

## Course Notes: Day 27

### Geometry of Cubic Bezier & Hermite Patches

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Today, we want to look at the geometry of placing the control points for bicubic Bezier patches. Specifically, how does the positions of the control points shape the surface. In explaining this, we will look at the generic structure of a bicubic Hermite patch. First, we develop formulas for the derivatives of Bezier patches. From this, we can show that knowing the first derivative vectors and the twist vector at each of the corner points of a patch allows you to construct the patch. This gives the Hermite form of a bicubic patch, rather than the Bezier form of a bicubic patch. The twist vector in essence is the mixed partial derivative of the vector valued function  $X(u, v)$ .

#### 27.1 Derivatives of Bezier patches

Given a control net for a Bezier patch (a grid of control points)  $p_{i,j}$ . The Bezier patch is given by

$$X(u, v) = \sum_{i=0}^m \sum_{j=0}^n \beta_i^m(u) \beta_j^n(v) p_{i,j}.$$

We can take the partial derivatives with respect to  $u$  and  $v$  to get

$$\frac{\partial X}{\partial u} = \sum_{i=0}^m \sum_{j=0}^n (\beta_i^m(u))' \beta_j^n(v) p_{i,j}$$

and

$$\frac{\partial X}{\partial v} = \sum_{i=0}^m \sum_{j=0}^n \beta_i^m(u) (\beta_j^n(v))' p_{i,j}.$$

Recalling the relations between the derivatives of the Bernstein polynomials,

$$\begin{aligned} \frac{d}{dt} (\beta_i^n(t)) &= \binom{n}{i} (i t^{i-1} (1-t)^{n-i} - (n-i) t^i (1-t)^{n-i-1}) \\ &= n \binom{n-1}{i-1} t^{i-1} (1-t)^{n-i} - n \binom{n-1}{i} t^i (1-t)^{n-i-1} \\ &= n [\beta_{i-1}^{n-1}(t) - \beta_i^{n-1}(t)], \end{aligned}$$

we have

$$\frac{\partial X}{\partial u} = \sum_{i=0}^m \sum_{j=0}^n m [\beta_{i-1}^{m-1}(u) - \beta_i^{m-1}(u)] \beta_j^n(v) p_{i,j}$$

and

$$\frac{\partial X}{\partial u} = \sum_{i=0}^m \sum_{j=0}^n n \beta_i^m(u) [\beta_{j-1}^{n-1}(v) - \beta_j^{n-1}(v)] p_{i,j}.$$

This gives a formal method for differentiating a Bezier patch from the functional representation of the Bezier patch.

We can express these forms of the derivatives in terms of first differences of the control net. The difference being that we have two first differences, one for each direction in the grid. We have a first difference in the  $u$ -direction (the  $i$  index)

$$\Delta_{1,0}p_{i,j} = p_{i+1,j} - p_{i,j},$$

and a first difference in the  $v$ -direction (the  $j$  index)

$$\Delta_{0,1}p_{i,j} = p_{i,j+1} - p_{i,j}.$$

By rearranging the sums, we have

$$\frac{\partial P}{\partial u} = \sum_{i=0}^{m-1} \sum_{j=0}^n m \beta_i^{m-1}(u) \beta_j^n(v) \Delta_{1,0}p_{i,j}$$

and

$$\frac{\partial X}{\partial u} = \sum_{i=0}^m \sum_{j=0}^n n \beta_i^m(u) \beta_j^{n-1}(v) \Delta_{0,1}p_{i,j}.$$

Notice that each derivative is also a Bezier patch. The grid on which the Bezier patch is defined is smaller and consists of vectors, as for Bezier curves.

To take the second derivatives of a Bezier patch, we proceed in a similar manner. As with Bezier curves, since the derivative of a Bezier patch is a Bezier patch, we can use the above methods to define the second derivatives. The second derivatives are

$$\frac{\partial^2 X}{\partial u^2} = \sum_{i=0}^{m-2} \sum_{j=0}^n m(m-1) \beta_i^{m-2}(u) \beta_j^n(v) \Delta_{2,0}p_{i,j}$$

and

$$\frac{\partial^2 X}{\partial u \partial v} = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} mn \beta_i^{m-1}(u) \beta_j^{n-1}(v) \Delta_{1,1}p_{i,j}$$

and

$$\frac{\partial^2 X}{\partial v^2} = \sum_{i=0}^m \sum_{j=0}^{n-2} n(n-1) \beta_i^m(u) \beta_j^{n-2}(v) \Delta_{0,2}p_{i,j}$$

where  $\Delta_{k,l}p_{i,j}$  are the appropriate differences. The differences above are given by

$$\begin{aligned} \Delta_{2,0}p_{i,j} &= \Delta_{1,0}p_{i+1,j} - \Delta_{1,0}p_{i,j} \\ &= p_{i+2,j} - 2p_{i+1,j} + p_{i,j} \end{aligned}$$

and

$$\begin{aligned} \Delta_{1,1}p_{i,j} &= \Delta_{1,0}p_{i,j+1} - \Delta_{1,0}p_{i,j} \\ &= \Delta_{0,1}p_{i+1,j} - \Delta_{0,1}p_{i,j} \\ &= p_{i+1,j+1} - p_{i,j+1} - p_{i+1,j} + p_{i,j} \end{aligned}$$

and

$$\begin{aligned} \Delta_{0,2}p_{i,j} &= \Delta_{0,1}p_{i,j+1} - \Delta_{0,1}p_{i,j} \\ &= p_{i,j+2} - 2p_{i,j+1} + p_{i,j}. \end{aligned}$$

We can proceed to take higher derivatives in a similar manner.

Figure 1: AAA

## 27.2 Derivatives of Bezier Patches at Corner Points

Let's look at the above formulas at corner points  $(u = 0, v = 0)$ ,  $(u = 0, v = 1)$ ,  $(u = 1, v = 0)$ ,  $(u = 1, v = 1)$ . The important facts to remember are that  $\beta_0^M(0) = 1$ ,  $\beta_i^M(0) = 0$  for  $i > 0$ , and  $\beta_M^M(1) = 1$ ,  $\beta_i^M(1) = 0$  for  $i < M$ . This means, that at the corner  $(u = 0, v = 0)$  we have

$$\begin{aligned}\frac{\partial X}{\partial u}\Big|_{(0,0)} &= m(p_{1,0} - p_{0,0}) \\ \frac{\partial X}{\partial v}\Big|_{(0,0)} &= n(p_{0,1} - p_{0,0}) \\ \frac{\partial^2 X}{\partial u^2}\Big|_{(0,0)} &= m(m-1)(p_{2,0} - 2p_{1,0} + p_{0,0}) \\ \frac{\partial^2 X}{\partial u \partial v}\Big|_{(0,0)} &= mn(p_{1,1} - p_{1,0} - p_{0,1} + p_{0,0}) \\ \frac{\partial^2 X}{\partial v^2}\Big|_{(0,0)} &= n(n-1)(p_{0,2} - 2p_{0,1} + p_{0,0}).\end{aligned}\tag{27.2.1}$$

The other corners are handled similarly.

The observation from (27.2.1) is that knowing the corner point  $X(0,0) = p_{0,0}$ , the first partial derivatives at the corner, and the mixed partial derivative at the corner, it is possible to determine the control points  $p_{1,0}$ ,  $p_{0,1}$  and  $p_{1,1}$ . The same manipulations can be done at the other corners, and thus if you know at each corner point of patch, the first partial derivatives (two vectors), and the mixed partial derivative (the twist vector) you can reconstruct a bicubic Bezier patch. When you give this information, you have a bicubic Hermite patch. This is the analogue of a Hermite curve.

## 27.3 Geometric Structure of the Control Net for a Bicubic Bezier Patch

The above derivations provide the geometric structure of the control points. The control points  $p_{0,0}$ ,  $p_{1,0}$  and  $p_{0,1}$  specify the tangent plane to the patch at the point  $p_{0,0}$ . Moreover, the vectors  $p_{1,0} - p_{0,0}$  and  $p_{0,1} - p_{0,0}$  control the tangent vectors of the *edges* of the patch at  $p_{0,0}$ , see diagram below.

The geometric meaning of the *twist vector* is harder to understand. First, let's consider why the mixed partial derivative of a parametric surface is called a twist vector. The mixed partial derivative at  $u = 0$ ,  $v = 0$  is the rate of change of the tangent vectors  $\frac{\partial X}{\partial u}\Big|_{(0,v)}$  as  $v$  varies and the rate of change of the tangent vectors  $\frac{\partial X}{\partial v}\Big|_{(u,0)}$  as  $u$  varies. This is called a twist vector because when the mixed partial derivative is zero the tangent plane at  $(0,0)$  is

Figure 2: Twist Vectors and Twisting of Tangent Planes

essentially given by either the pair of lines

$$\begin{aligned} X(0,0) + s \frac{\partial X}{\partial u} \Big|_{0,0} \\ X(0,v) + s \frac{\partial X}{\partial u} \Big|_{0,v} \end{aligned}$$

or

$$\begin{aligned} X(0,0) + s \frac{\partial X}{\partial v} \Big|_{0,0} \\ X(u,0) + s \frac{\partial X}{\partial v} \Big|_{u,0} \end{aligned}$$

for small values of  $u$  and  $v$ , as both pair of lines are essentially parallel. Moreover, when the twist vector,  $\partial^2 X / \partial u \partial v$ , is nonzero, it is the vector  $\partial^2 X / \partial u \partial v \Big|_{(0,0)}$  is the offset from the lines being parallel, that is the lines

$$\begin{aligned} X(0,0) + s \frac{\partial X}{\partial u} \Big|_{(0,0)} \\ X(0,v) + s \left( \frac{\partial X}{\partial u} \Big|_{(0,v)} - v \frac{\partial^2 X}{\partial u} \partial v \Big|_{(0,0)} \right) \end{aligned}$$

and the lines

$$\begin{aligned} X(0,0) + s \frac{\partial X}{\partial v} \Big|_{(0,0)} \\ X(u,0) + s \left( \frac{\partial X}{\partial v} \Big|_{(u,0)} - u \frac{\partial^2 X}{\partial u} \partial v \Big|_{(0,0)} \right) \end{aligned}$$

are essentially parallel and give the tangent plane. This represents a twisting of the tangent planes along the curves  $X(u,0)$  and  $X(0,v)$ , see diagram below.

For a bicubic Bezier patch, the twist vectors will determine the interior control points  $p_{1,1}$ ,  $p_{1,2}$ ,  $p_{2,1}$  and  $p_{2,2}$ . The effect of specifying a twist vector is not entirely obvious. Instead, it is more standard to specify a zero twist vector and shape a bicubic Bezier patch by specifying planes at the four corners, as a zero twist vector implies that the control points  $p_{0,0}$ ,  $p_{1,0}$ ,  $p_{0,1}$  and  $p_{1,1}$  are coplanar.

## Exercises

In the majority of these exercises, you are to work on understanding the Bezier patches, particularly bicubic Bezier patches. The goal being to understand the effect of placing control points, and develop a geometric intuition of the effect of each control point on the shape of the bicubic Bezier patch. To accomplish this, you are to use the file **bpatch.txt** and the file **bpatch.mws** available on the ANGEL, together with the interactive applets for creating Bezier patches and Bezier patches with zero twist vectors..

1. Start with the control net

$$\begin{bmatrix} p_{0,0} = [0, 0, 0] & p_{1,0} = [1, 0, 0] & p_{2,0} = [2, 0, 0] & p_{3,0} = [3, 0, 0] \\ p_{0,1} = [0, 1, 0] & p_{1,1} = [1, 1, 0] & p_{2,1} = [2, 1, 0] & p_{3,1} = [3, 1, 0] \\ p_{0,2} = [0, 2, 0] & p_{1,2} = [1, 2, 0] & p_{2,2} = [2, 2, 0] & p_{3,2} = [3, 2, 0] \\ p_{0,3} = [0, 3, 0] & p_{1,3} = [1, 3, 0] & p_{2,3} = [2, 3, 0] & p_{3,3} = [3, 3, 0] \end{bmatrix}$$

Leave the exterior control point alone, and change the interior control points  $p_{1,1}$ ,  $p_{1,2}$ ,  $p_{2,1}$ ,  $p_{2,2}$ . How does the surface change?

- (a) First change only the  $z$  component.
- (b) Compute the Bezier patch using the command **BPatch**.
- (c) Plot the patch using the command **PatchPlot**
- (d) Plot the control net using the command **ContNet**
- (e) Display both the patch and the control net together.
- (f) Next change every component.

2. Start with the control net

$$\begin{bmatrix} p_{0,0} = [0, 0, 0] & p_{1,0} = [1, 0, 0] & p_{2,0} = [2, 0, 0] & p_{3,0} = [3, 0, 0] \\ p_{0,1} = [0, 1, 0] & p_{1,1} = [1, 1, 0] & p_{2,1} = [2, 1, 0] & p_{3,1} = [3, 1, 0] \\ p_{0,2} = [0, 2, 0] & p_{1,2} = [1, 2, 0] & p_{2,2} = [2, 2, 0] & p_{3,2} = [3, 2, 0] \\ p_{0,3} = [0, 3, 0] & p_{1,3} = [1, 3, 0] & p_{2,3} = [2, 3, 0] & p_{3,3} = [3, 3, 0] \end{bmatrix}$$

Leave the interior control points alone and change the exterior control points  $p_{0,0}$ ,  $p_{1,0}$ ,  $p_{2,0}$ ,  $p_{3,0}$ ,  $p_{0,1}$ ,  $p_{3,1}$ ,  $p_{0,2}$ ,  $p_{3,2}$ ,  $p_{0,3}$ ,  $p_{1,3}$ ,  $p_{2,3}$ ,  $p_{3,3}$ . How does the surface change?

- (a) First change only the  $z$  component.
- (b) Compute the Bezier patch using the command **BPatch**.
- (c) Plot the patch using the command **PatchPlot**
- (d) Plot the control net using the command **ContNet**
- (e) Display both the patch and the control net together.
- (f) Next change every component. [In particular what happens, if the some of the exterior control points are the same.]

3. Create a general catalog of bicubic Bezier patches.
4. Create a general catalog of bicubic Bezier patches with zero twist vectors.
5. **CONTEST**: Create the most interesting bicubic Bezier patch you can. Think about the placement of the control points. You will need to be able to say why you think this is an interesting patch. The most interesting bicubic Bezier patches as voted in class tomorrow will earn extra credit points.
6. Show that a zero twist vector implies that the four control points  $p_{0,0}$ ,  $p_{1,0}$ ,  $p_{0,1}$  and  $p_{1,1}$  are coplanar.